Relativistic Lagrangians for the Lorentz-Dirac equation

Shinichi Deguchi[a,*], Kunihiko Nakano[a], Takafumi Suzuki[b]

[a]*Institute of Quantum Science, College of Science and Technology, Nihon University, Chiyoda-ku, Tokyo 101-8308, Japan*
[b]*Junior College Funabashi Campus, Nihon University, Narashinodai, Funabashi, Chiba 274-8501, Japan*

Abstract

We present two types of relativistic Lagrangians for the Lorentz-Dirac equation written in terms of an arbitrary world-line parameter. One of the Lagrangians contains an exponential damping function of the proper time and explicitly depends on the world-line parameter. Another Lagrangian includes additional cross-terms consisting of auxiliary dynamical variables and does not depend explicitly on the world-line parameter. We demonstrate that both the Lagrangians actually yield the Lorentz-Dirac equation with a source-like term.

1. Introduction

A charged particle emitting electromagnetic radiation is subjected to the reaction force caused by the particle's own electromagnetic radiation. This phenomenon is well-known as the *radiation reaction* [1, 2, 3, 4, 5, 6]. It was first evaluated by Lorentz at the end of the 19th century [7] and subsequently argued by Abraham and Lorentz on the basis of the charged rigid sphere model of a charged particle [8, 9]. In the zero radius limit that this model tends to become a point charge, the classical non-relativistic equation of motion for the charged particle located at the position $\boldsymbol{x} = \boldsymbol{x}(t)$ is found to be

$$m\frac{d^2\boldsymbol{x}}{dt^2} = \boldsymbol{F} + \frac{2}{3}e^2\frac{d^3\boldsymbol{x}}{dt^3}, \tag{1.1}$$

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*Corresponding author.
Email address: `deguchi@phys.cst.nihon-u.ac.jp` (Shinichi Deguchi)

where m is the physical mass of the particle, e its electric charge, and $\boldsymbol{F}$ denotes the external Lorentz force. (In this paper, we employ units such that $c = 1$.) Equation (1.1) is called the Lorentz-Abraham equation (or the Abraham-Lorentz equation). A relativistic extension of the Lorentz-Abraham equation was derived by Dirac in a manifestly covariant manner by considering energy-momentum conservation [10], and is now often called the Lorentz-Dirac equation [3, 6, 11, 12, 13]. With the spacetime coordinates $x^\mu = x^\mu(l)$ ($\mu = 0, 1, 2, 3$) of a charged particle propagating in 4-dimensional Minkowski space, the Lorentz-Dirac equation reads

$$m\frac{du^\mu}{dl} = eF^{\mu\nu}(x)u_\nu + \frac{2}{3}e^2\big(\delta^\mu{}_\nu - u^\mu u_\nu\big)\frac{d^2u^\nu}{dl^2}\,. \qquad (1.2)$$

Here, $u^\mu := dx^\mu/dl$, $F^{\mu\nu}$ is the field strength tensor of an external electromagnetic field, and l denotes the proper time of the particle or, in other words, the arc length of the world-line traced out by the particle. The metric tensor of Minkowski space is assumed to be $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, so that $u_\mu u^\mu = \eta_{\mu\nu}u^\mu u^\nu = 1$ holds. Equations (1.1) and (1.2) are unusual ones including third-order time derivatives of the particle's position coordinates. In connection with this fact, these equations admit physically unacceptable solutions such as runaway and pre-acceleration solutions [1, 2, 4, 5, 6, 12, 13, 14]. To overcome this problem, various ideas have been proposed until recently [1, 6, 13, 14, 15, 16, 17, 18, 19, 20, 21]; however, it seems that an ultimate solution to the problem has not been found yet.

Once the equations of motion (1.1) and (1.2) have been obtained, it is quite natural to seek Lagrangians corresponding to these equations in order to develop the Lagrangian and Hamiltonian formulations of a charged particle subjected to the radiation reaction force. If these formulations are established, they might lead to a novel quantum-mechanical description of a charged particle undergoing radiation reaction and might give us new room to deal with the above-mentioned problem. As far as the present authors know, there have been a few attempts to construct Lagrangians corresponding to Eqs. (1.1) and (1.2) until now [22, 23, 24]. Carati constructed an explicitly time-dependent Lagrangian for Eq (1.1) with the use of auxiliary dynamical variables [22]. (Carati also considered a relativistic extension of this Lagrangian in an extremely limited case.) Barone and Mendes derived an explicitly time-independent Lagrangian for Eq. (1.1) by incorporating the time-reversed copy of Eq. (1.1) into the original setting [23]. Carati's and Barone-Mendes's approaches are, respectively, based on learning from

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the direct and indirect Lagrangian formulations of the damped harmonic oscillator [25, 26, 27, 28, 29, 30].[1] It should be pointed out here that in these approaches, the external Lorentz force $\boldsymbol{F}$ is assumed to be independent of the velocity $\boldsymbol{v} := d\boldsymbol{x}/dt$. Hence, it follows that in actuality, Carati's and Barone-Mendes's Lagrangians can describe only a charged particle being in the particular situation in which the magnetic field vanishes or is parallel to $\boldsymbol{v}$.[2]

In this paper, we present two types of Lagrangians for the Lorentz-Dirac equation (1.2) that are constructed in such a fashion that the corresponding actions remain invariant under reparametrization of a world-line parameter along the particle's world-line. These Lagrangians are completely relativistic and admit the general form of the external Lorentz force. Also, the Lagrangians are outside the scope of Kupriyanov's proof [24], because they contain auxiliary dynamical variables in addition to x^μ. One of the Lagrangians contains an exponential damping function of the proper time l, while another Lagrangian includes additional cross-terms consisting of two auxiliary dynamical variables. Both the Lagrangians include terms similar to what can be seen in the Lagrangian that governs a certain model of a relativistic point particle with rigidity [31, 32, 33]. We would like to emphasize that our Lagrangians are not immediate extensions of Carati's and Barone-Mendes's Lagrangians.

This paper is organized as follows. In section 2, we introduce necessary dynamical variables and define their transformation rules under reparametrization of a world-line parameter. In section 3, we present a Lagrangian that contains an exponential damping function and show that the Lagrangian actually yields the Lorentz-Dirac equation with a source-like term. In section 4, we consider a Lagrangian including additional cross-terms, instead of the

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[1] The direct formulation adopts an explicitly time-dependent Lagrangian of the damped harmonic oscillator [25, 26, 27], while the indirect formulation adopts an explicitly time-independent Lagrangian for a system consisting of the damped harmonic oscillator and its time-reversed counterpart [25, 28, 29, 30].

[2] In Ref. [24], Kupriyanov investigated the possibility of constructing Lagrangians corresponding to Eqs. (1.1) and (1.2) and reached the conclusion that there exist no corresponding Lagrangians. However, Kupriyanov's proof of this conclusion considers Lagrangians consisting only of the coordinate variables, such as $\boldsymbol{x}$ and x^μ, and their first- and second-order time derivatives. Since Carati's and Barone-Mendes's Lagrangians contain extra dynamical variables, these Lagrangians are outside the scope of Kupriyanov's proof.

exponential damping function, and show that this Lagrangian also yields the Lorentz-Dirac equation with a source-like term. Section 5 is devoted to a summary and discussion. Appendix A provides the Lorentz-Dirac equation written in terms of an arbitrary world-line parameter instead of the proper time l.

2. Preliminaries: dynamical variables and their transformation rules

Let τ ($\tau_0 \leq \tau \leq \tau_1$) be an arbitrary world-line parameter along the particle's world-line, being chosen in such a manner that $dx^0/d\tau > 0$. The spacetime coordinates of a charged particle are now denoted as $x^\mu = x^\mu(\tau)$. Under the reparametrization $\tau \to \tau' = \tau'(\tau)$ ($d\tau'/d\tau > 0$), the coordinate variables x^μ behave as scalar fields on the 1-dimensional parameter space $\mathcal{T} := \{\tau \,|\, \tau_0 \leq \tau \leq \tau_1\}$:

$$x^\mu(\tau) \to x'^\mu(\tau') = x^\mu(\tau). \tag{2.1}$$

In addition to x^μ, we introduce auxiliary dynamical variables $q_i^\mu = q_i^\mu(\tau)$, $\lambda_{i\mu} = \lambda_{i\mu}(\tau)$ ($i = 1, 2$), and $\xi_\mu = \xi_\mu(\tau)$. They are assumed to transform under the reparametrization as scalar-density fields of weight 1 on $\mathcal{T}$:

$$q_i^\mu(\tau) \to q_i'^\mu(\tau') = \frac{d\tau}{d\tau'} q_i^\mu(\tau), \tag{2.2}$$

$$\lambda_{i\mu}(\tau) \to \lambda'_{i\mu}(\tau') = \frac{d\tau}{d\tau'} \lambda_{i\mu}(\tau), \tag{2.3}$$

$$\xi_\mu(\tau) \to \xi'_\mu(\tau') = \frac{d\tau}{d\tau'} \xi_\mu(\tau). \tag{2.4}$$

The components of the vector resolute of the 4-vector $(\dot{q}_i^\mu)$ perpendicular to (q_i^μ) are given by

$$\dot{q}_{i\perp}^\mu := \dot{q}_i^\mu - \frac{q_i \dot{q}_i}{q_i^2} q_i^\mu, \tag{2.5}$$

where $\dot{q}_i^\mu := dq_i^\mu/d\tau$, $q_i^2 := q_{i\mu} q_i^\mu$, and $q_i \dot{q}_i := q_{i\mu} \dot{q}_i^\mu$ (no sum with respect to i). It can be shown by using Eq. (2.2) that unlike $\dot{q}_i^\mu$, the components $\dot{q}_{i\perp}^\mu$ transform homogeneously as

$$\dot{q}_{i\perp}^\mu(\tau) \to \dot{q}_{i\perp}'^\mu(\tau') = \left(\frac{d\tau}{d\tau'}\right)^2 \dot{q}_{i\perp}^\mu(\tau). \tag{2.6}$$

We thus see that under the reparametrization, $\dot{q}_{i\perp}^\mu$ behave as scalar-density fields of weight 2 on $\mathcal{T}$.

3. A Lagrangian with an exponential damping function

Now, from the dynamical variables x^μ, q_i^μ, $\lambda_{i\mu}$, and ξ_μ, we construct the following Lagrangian:

$$L_\mathrm{D} = \frac{\exp(-kl)}{(q_1^2 q_2^2)^{1/4}} \left[\frac{1}{2} \left(\frac{\dot{q}_{1\perp}^2}{q_1^2} - \frac{\dot{q}_{2\perp}^2}{q_2^2} \right) \right.$$

$$\left. - \lambda_{1\mu} (q_1^\mu - \dot{x}^\mu) + \lambda_{2\mu} (q_2^\mu - \dot{x}^\mu) + \xi_\mu (q_1^\mu - q_2^\mu) - \frac{3}{2e} F_{\mu\nu}(x) q_1^\mu q_2^\nu \right],$$

$$(3.1)$$

where $k := 3m/2e^2$, $\dot{x}^\mu := dx^\mu/d\tau$, $\dot{q}_{i\perp}^2 := \dot{q}_{i\perp\mu} \dot{q}_{i\perp}^\mu$, and $F_{\mu\nu}(= -F_{\nu\mu})$ is again the field strength tensor of an external electromagnetic field. In Eq. (3.1), the proper time l is a function of τ represented as

$$l(\tau) = \int_{\tau_0}^\tau d\tilde{\tau} \sqrt{\dot{x}_\mu(\tilde{\tau}) \dot{x}^\mu(\tilde{\tau})} . \qquad (3.2)$$

Here, $x^\mu(\tilde{\tau})$ is understood as a solution of the equation of motion for x^μ obtained later, not as a dynamical variable whose variation is taken into account in varying the action

$$S_\mathrm{D} = \int_{\tau_0}^{\tau_1} d\tau\, L_\mathrm{D} . \qquad (3.3)$$

The Lagrangian L_D explicitly depends on τ via the exponential damping function $\exp(-kl)$. Since $l(\tau)$ is geometrically the arc length of the particle's world-line, it is certainly reparametrization invariant.[3] Considering this fact and using the transformation rules in Eqs. (2.1), (2.2), (2.3), (2.4), and (2.6), we can show that the action S_D remains invariant under the reparametrization $\tau \to \tau'$. We also see that L_D remains invariant under the gauge transformation

$$\lambda_{1\mu} \to \lambda'_{1\mu} = \lambda_{1\mu} + \theta_\mu , \quad \lambda_{2\mu} \to \lambda'_{2\mu} = \lambda_{2\mu} + \theta_\mu , \quad \xi_\mu \to \xi'_\mu = \xi_\mu + \theta_\mu ,$$

$$(3.4)$$

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[3] Strictly speaking, $l(\tau)$ is a functional of x^μ as well as a function of τ and τ_0. In this sense, $l(\tau)$ should be read as $l(\tau, \tau_0; x^\mu)$. The reparametrization invariance of $l(\tau)$ can be expressed as $l(\tau', \tau'_0; x'^\mu) = l(\tau, \tau_0; x^\mu)$.

with real gauge functions $\theta^\mu = \theta^\mu(\tau)$. The Lagrangian L_D has the antisymmetric property

$$L_D\left(q_1^\mu, \dot{q}_1^\mu, \lambda_{1\mu}; q_2^\mu, \dot{q}_2^\mu, \lambda_{2\mu}\right) = -L_D\left(q_2^\mu, \dot{q}_2^\mu, \lambda_{2\mu}; q_1^\mu, \dot{q}_1^\mu, \lambda_{1\mu}\right). \qquad (3.5)$$

Let us derive the Euler-Lagrange equations for the dynamical variables from L_D. Noting that $x^\mu(\tilde{\tau})$ contained in $l(\tau)$ and hence $l(\tau)$ itself are not objects for taking variation, we can easily obtain the Euler-Lagrange equation for x^μ:

$$\frac{d}{d\tau}\left[\frac{\exp(-kl)}{\left(q_1^2 q_2^2\right)^{1/4}}\left(\lambda_{1\mu} - \lambda_{2\mu}\right)\right] + \frac{3\exp(-kl)}{2e\left(q_1^2 q_2^2\right)^{1/4}}\partial_\mu F_{\nu\rho}(x) q_1^\nu q_2^\rho = 0. \qquad (3.6)$$

This equation includes the gauge-invariant quantity $\lambda_{1\mu} - \lambda_{2\mu}$ as a reflection of the gauge invariance of L_D. Hence, $\lambda_{1\mu}$ and $\lambda_{2\mu}$ themselves are not uniquely determined. The Euler-Lagrange equation for q_1^μ can be written as

$$\frac{\exp(-kl)}{\left(q_1^2 q_2^2\right)^{1/4}}\left[\frac{1}{2}\left(\frac{d}{d\tau}\frac{\partial K_1}{\partial \dot{q}_1^\mu} - \frac{\partial K_1}{\partial q_1^\mu}\right) + \lambda_{1\mu} - \xi_\mu + \frac{3}{2e}F_{\mu\nu}(x)q_2^\nu\right]$$

$$+\left(\frac{d}{d\tau}\frac{\exp(-kl)}{\left(q_1^2 q_2^2\right)^{1/4}}\right)\frac{1}{2}\frac{\partial K_1}{\partial \dot{q}_1^\mu} + \frac{q_{1\mu}}{2q_1^2}L_D = 0, \qquad (3.7)$$

with

$$K_1 := \frac{\dot{q}_{1\perp}^2}{q_1^2} = \frac{q_1^2 \dot{q}_1^2 - (q_1 \dot{q}_1)^2}{\left(q_1^2\right)^2}. \qquad (3.8)$$

Applying the formulas

$$\frac{1}{2}\frac{\partial K_1}{\partial \dot{q}_1^\mu} = \frac{\dot{q}_{1\perp\mu}}{q_1^2} = \frac{1}{\sqrt{q_1^2}}\frac{d}{d\tau}\frac{q_{1\mu}}{\sqrt{q_1^2}}, \qquad (3.9)$$

$$\frac{d}{d\tau}\frac{\partial K_1}{\partial \dot{q}_1^\mu} - \frac{\partial K_1}{\partial q_1^\mu} = \frac{2}{q_1^2}\left(\ddot{q}_{1\perp\mu} - \frac{2q_1 \dot{q}_1}{q_1^2}\dot{q}_{1\perp\mu}\right), \qquad (3.10)$$

$$\frac{d}{d\tau}\frac{1}{\left(q_1^2 q_2^2\right)^{1/4}} = -\frac{1}{2\left(q_1^2 q_2^2\right)^{1/4}}\left(\frac{q_1 \dot{q}_1}{q_1^2} + \frac{q_2 \dot{q}_2}{q_2^2}\right) \qquad (3.11)$$

to Eq. (3.7) appropriately, we obtain

$$k\frac{dl}{d\tau}\frac{1}{\sqrt{q_1^2}}\frac{d}{d\tau}\frac{q_{1\mu}}{\sqrt{q_1^2}} = \frac{3}{2e}F_{\mu\nu}(x)q_2^\nu + \frac{\left(q_1^2 q_2^2\right)^{1/4} q_{1\mu}}{2q_1^2 \exp(-kl)}L_D$$

$$+ \frac{\ddot{q}_{1\perp\mu}}{q_1^2} - \left(\frac{5q_1 \dot{q}_1}{q_1^2} + \frac{q_2 \dot{q}_2}{q_2^2}\right)\frac{\dot{q}_{1\perp\mu}}{2q_1^2} + \lambda_{1\mu} - \xi_\mu. \qquad (3.12)$$

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Here, $\ddot{q}_{1\perp\mu}$, together with $\ddot{q}_{2\perp\mu}$, is defined by

$$\ddot{q}_{i\perp\mu} := \ddot{q}_{i\mu} - \frac{q_i \ddot{q}_i}{q_i^2} q_{i\mu}, \tag{3.13}$$

where $\ddot{q}_{i\mu} := d^2 q_{i\mu}/d\tau^2$ and $q_i \ddot{q}_i := q_{i\mu} \ddot{q}_i^\mu$ (no sum with respect to i). Following the same procedure as that used for deriving Eq. (3.12), we can derive the Euler-Lagrange equation for q_2^μ as

$$k \frac{dl}{d\tau} \frac{1}{\sqrt{q_2^2}} \frac{d}{d\tau} \frac{q_{2\mu}}{\sqrt{q_2^2}} = \frac{3}{2e} F_{\mu\nu}(x) q_1^\nu - \frac{(q_1^2 q_2^2)^{1/4}}{2 q_2^2 \exp(-kl)} \frac{q_{2\mu}}{L_D}$$
$$+ \frac{\ddot{q}_{2\perp\mu}}{q_2^2} - \left(\frac{q_1 \dot{q}_1}{q_1^2} + \frac{5 q_2 \dot{q}_2}{q_2^2} \right) \frac{\dot{q}_{2\perp\mu}}{2 q_2^2} + \lambda_{2\mu} - \xi_\mu. \tag{3.14}$$

The Euler-Lagrange equations for $\lambda_{1\mu}$, $\lambda_{2\mu}$, and ξ_μ are respectively found to be

$$q_1^\mu = \dot{x}^\mu, \tag{3.15}$$
$$q_2^\mu = \dot{x}^\mu, \tag{3.16}$$
$$q_1^\mu = q_2^\mu. \tag{3.17}$$

Equation (3.17) can also be found from Eqs. (3.15) and (3.16).

Substituting Eqs. (3.15) and (3.16) into Eq. (3.12) and noting

$$L_D \left(q_1^\mu, \dot{q}_1^\mu, \lambda_{1\mu}; q_2^\mu, \dot{q}_2^\mu, \lambda_{2\mu} \right) = L_D \left(\dot{x}^\mu, \ddot{x}^\mu, \lambda_{1\mu}; \dot{x}^\mu, \ddot{x}^\mu, \lambda_{2\mu} \right) = 0, \tag{3.18}$$

we have

$$k \frac{dl}{d\tau} \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{d\tau} \frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} = \frac{3}{2e} F^{\mu\nu}(x) \dot{x}_\nu + \frac{\dddot{x}_\perp^\mu}{\dot{x}^2} - \frac{3(\dot{x}\ddot{x})\ddot{x}_\perp^\mu}{(\dot{x}^2)^2} + \lambda_1^\mu - \xi^\mu, \tag{3.19}$$

where $\ddot{x}^\mu := d^2 x^\mu/d\tau^2$ and $\dddot{x}^\mu := d^3 x^\mu/d\tau^3$. Similarly, substituting Eqs. (3.15) and (3.16) into Eq. (3.14) and using (3.18), we have

$$k \frac{dl}{d\tau} \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{d\tau} \frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} = \frac{3}{2e} F^{\mu\nu}(x) \dot{x}_\nu + \frac{\dddot{x}_\perp^\mu}{\dot{x}^2} - \frac{3(\dot{x}\ddot{x})\ddot{x}_\perp^\mu}{(\dot{x}^2)^2} + \lambda_2^\mu - \xi^\mu. \tag{3.20}$$

Comparing Eq. (3.19) with Eq. (3.20) leads to

$$\lambda_1^\mu = \lambda_2^\mu. \tag{3.21}$$

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This equality is covariant under the gauge transformation (3.4). It follows from Eq. (3.21) that Eq. (3.6) is identically satisfied, because $\partial_\mu F_{\nu\rho}(x)q_1^\nu q_2^\rho = \partial_\mu F_{\nu\rho}(x)\dot{x}^\nu \dot{x}^\rho = 0$ holds by using Eqs. (3.15) and (3.16). Hereafter, taking into account Eq. (3.21), we simply write λ_1^μ and λ_2^μ as λ^μ. Thereby, Eqs (3.19) and (3.20) can be written together as a single equation

$$k\frac{dl}{d\tau}\frac{1}{\sqrt{\dot{x}^2}}\frac{d}{d\tau}\frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} = \frac{3}{2e}F^{\mu\nu}(x)\dot{x}_\nu + \frac{\dddot{x}_\perp^\mu}{\dot{x}^2} - \frac{3(\dot{x}\ddot{x})\ddot{x}_\perp^\mu}{(\dot{x}^2)^2} + \Lambda^\mu, \qquad (3.22)$$

where $\Lambda^\mu := \lambda^\mu - \xi^\mu$. Obviously, Λ^μ is invariant under the gauge transformation (3.4). Equation (3.22) is precisely the equation of motion for x^μ mentioned under Eq. (3.2). Since x^μ contained in l has been assumed to be a solution of Eq. (3.22), it can be identified with x^μ in Eq. (3.22). Upon considering this fact, substituting the τ-derivative of Eq. (3.2), i.e., $dl(\tau)/d\tau = \sqrt{\dot{x}_\mu(\tau)\dot{x}^\mu(\tau)}$, into Eq. (3.22) and recalling $k := 3m/2e^2$, we obtain

$$m\frac{d}{d\tau}\frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} = eF^{\mu\nu}(x)\dot{x}_\nu + \frac{2}{3}e^2\left(\frac{\dddot{x}_\perp^\mu}{\dot{x}^2} - \frac{3(\dot{x}\ddot{x})\ddot{x}_\perp^\mu}{(\dot{x}^2)^2}\right) + \frac{2}{3}e^2\Lambda^\mu. \qquad (3.23)$$

If $\Lambda^\mu = 0$, Eq. (3.23) is identical with the Lorentz-Dirac equation written in terms of the arbitrary world-line parameter τ; see Eq. (A.8) in Appendix A. For this reason, Eq. (3.23) can be said to be the Lorentz-Dirac equation with a source-*like* term $2e^2\Lambda^\mu/3$. We thus see that the Lagrangian L_{D} yields the Lorentz-Dirac equation with a source-like term.

Now we adopt the proper-time gauge $\tau = l$, choosing τ to be the proper time l. Accordingly, $\dot{x}^\mu = u^\mu$, $\dot{x}^2 = 1$, and $\dot{x}\ddot{x} = 0$ are valid, so that Eq. (3.23) becomes

$$m\frac{du^\mu}{dl} = eF^{\mu\nu}(x)u_\nu + \frac{2}{3}e^2\left(\delta^\mu{}_\nu - u^\mu u_\nu\right)\frac{d^2u^\nu}{dl^2} + \frac{2}{3}e^2\Lambda^\mu. \qquad (3.24)$$

This is exactly what is defined by adding the source-like term $2e^2\Lambda^\mu/3$ to the (original) Lorentz-Dirac equation (1.2).

4. A Lagrangian with additional cross-terms

Next we consider an alternative Lagrangian defined by

$$L_A = \frac{1}{(q_1^2 q_2^2)^{1/4}} \left[\frac{1}{2} \left(\frac{\dot{q}_{1\perp}^2}{q_1^2} - \frac{\dot{q}_{2\perp}^2}{q_2^2} \right) - \frac{k}{2} \left(\frac{\dot{q}_{1\perp\mu} q_2^\mu}{\sqrt{q_1^2}} - \frac{\dot{q}_{2\perp\mu} q_1^\mu}{\sqrt{q_2^2}} \right) \right.$$

$$\left. - \lambda_{1\mu} (q_1^\mu - \dot{x}^\mu) + \lambda_{2\mu} (q_2^\mu - \dot{x}^\mu) + \xi_\mu (q_1^\mu - q_2^\mu) - \frac{3}{2e} F_{\mu\nu}(x) q_1^\mu q_2^\nu \right]. \tag{4.1}$$

Here it should be emphasized that L_A includes the additional cross-terms proportional to k instead of the exponential damping function $\exp(-kl)$. Also, it is worth noting that unlike L_D, the Lagrangian L_A does not depend explicitly on τ. Using the transformation rules in Eqs. (2.1), (2.2), (2.3), (2.4), and (2.6), we can show that the action

$$S_A = \int_{\tau_0}^{\tau_1} d\tau \, L_A \tag{4.2}$$

remains invariant under the reparametrization $\tau \to \tau'$. Just like L_D, the Lagrangian L_A remains invariant under the gauge transformation (3.4) and possesses the antisymmetric property

$$L_A \left(q_1^\mu, \dot{q}_1^\mu, \lambda_{1\mu}; q_2^\mu, \dot{q}_2^\mu, \lambda_{2\mu} \right) = -L_A \left(q_2^\mu, \dot{q}_2^\mu, \lambda_{2\mu}; q_1^\mu, \dot{q}_1^\mu, \lambda_{1\mu} \right). \tag{4.3}$$

We now derive the Euler-Lagrange equations for the dynamical variables from L_A. The Euler-Lagrange equation for x^μ is found to be

$$\frac{d}{d\tau} \left[\frac{1}{(q_1^2 q_2^2)^{1/4}} (\lambda_{1\mu} - \lambda_{2\mu}) \right] + \frac{3}{2e (q_1^2 q_2^2)^{1/4}} \partial_\mu F_{\nu\rho}(x) q_1^\nu q_2^\rho = 0. \tag{4.4}$$

This equation includes the gauge-invariant combination $\lambda_{1\mu} - \lambda_{2\mu}$ owing to the gauge invariance of L_A, and hence cannot determine $\lambda_{1\mu}$ and $\lambda_{2\mu}$ uniquely.

The Euler-Lagrange equation for q_1^μ can be written as

$$\frac{1}{(q_1^2 q_2^2)^{1/4}} \left[\frac{1}{2} \left(\frac{d}{d\tau} \frac{\partial K_1}{\partial \dot{q}_1^\mu} - \frac{\partial K_1}{\partial q_1^\mu} \right) - \frac{k}{2} \left(\frac{d}{d\tau} \frac{\partial J}{\partial \dot{q}_1^\mu} - \frac{\partial J}{\partial q_1^\mu} \right) + \lambda_{1\mu} - \xi_\mu + \frac{3}{2e} F_{\mu\nu}(x) q_2^\nu \right]$$

$$+ \left(\frac{d}{d\tau} \frac{1}{(q_1^2 q_2^2)^{1/4}} \right) \left(\frac{1}{2} \frac{\partial K_1}{\partial \dot{q}_1^\mu} - \frac{k}{2} \frac{\partial J}{\partial \dot{q}_1^\mu} \right) + \frac{q_{1\mu}}{2q_1^2} L_A = 0, \tag{4.5}$$

where K_1 and J are given by Eq. (3.8) and

$$J := \frac{\dot{q}_{1\perp\mu}q_2^{\mu}}{\sqrt{q_1^2}} - \frac{\dot{q}_{2\perp\mu}q_1^{\mu}}{\sqrt{q_2^2}}. \tag{4.6}$$

Applying the formulas (3.9)–(3.11) and

$$\frac{\partial J}{\partial \dot{q}_1^{\mu}} = \frac{1}{\sqrt{q_1^2}} \left(q_{2\mu} - \frac{q_1 q_2}{q_1^2} q_{1\mu} \right), \tag{4.7}$$

$$\frac{d}{d\tau}\frac{\partial J}{\partial \dot{q}_1^{\mu}} - \frac{\partial J}{\partial q_1^{\mu}} = \frac{d}{d\tau}\frac{q_{2\mu}}{\sqrt{q_2^2}} + \frac{1}{\sqrt{q_1^2}} \left(\dot{q}_{2\mu} - \frac{q_1 \dot{q}_2}{q_1^2} q_{1\mu} \right) \tag{4.8}$$

to Eq. (4.5) appropriately, we obtain

$$\frac{k}{2}\left[\frac{d}{d\tau}\frac{q_{2\mu}}{\sqrt{q_2^2}} + \frac{1}{\sqrt{q_1^2}} \left(\dot{q}_{2\mu} - \frac{q_1 \dot{q}_2}{q_1^2} q_{1\mu} \right) \right]$$
$$= \frac{3}{2e}F_{\mu\nu}(x)q_2^{\nu} + \frac{(q_1^2 q_2^2)^{1/4} q_{1\mu}}{2q_1^2}L_{\mathrm{A}} + \frac{\ddot{q}_{1\perp\mu}}{q_1^2} - \left(\frac{5q_1\dot{q}_1}{q_1^2} + \frac{q_2\dot{q}_2}{q_2^2} \right)\frac{\dot{q}_{1\perp\mu}}{2q_1^2} + \lambda_{1\mu} - \xi_{\mu}$$
$$+ \frac{k}{4\sqrt{q_1^2}} \left(\frac{q_1\dot{q}_1}{q_1^2} + \frac{q_2\dot{q}_2}{q_2^2} \right)\left(q_{2\mu} - \frac{q_1 q_2}{q_1^2} q_{1\mu} \right), \tag{4.9}$$

where $q_1 q_2 := q_{1\mu}q_2^{\mu}$ and $q_1\dot{q}_2 := q_{1\mu}\dot{q}_2^{\mu}$. Similarly, the Euler-Lagrange equation for q_2^{μ} is derived as

$$\frac{k}{2}\left[\frac{d}{d\tau}\frac{q_{1\mu}}{\sqrt{q_1^2}} + \frac{1}{\sqrt{q_2^2}} \left(\dot{q}_{1\mu} - \frac{\dot{q}_1 q_2}{q_2^2} q_{2\mu} \right) \right]$$
$$= \frac{3}{2e}F_{\mu\nu}(x)q_1^{\nu} - \frac{(q_1^2 q_2^2)^{1/4} q_{2\mu}}{2q_2^2}L_{\mathrm{A}} + \frac{\ddot{q}_{2\perp\mu}}{q_2^2} - \left(\frac{q_1\dot{q}_1}{q_1^2} + \frac{5q_2\dot{q}_2}{q_2^2} \right)\frac{\dot{q}_{2\perp\mu}}{2q_2^2} + \lambda_{2\mu} - \xi_{\mu}$$
$$+ \frac{k}{4\sqrt{q_2^2}} \left(\frac{q_1\dot{q}_1}{q_1^2} + \frac{q_2\dot{q}_2}{q_2^2} \right)\left(q_{1\mu} - \frac{q_1 q_2}{q_2^2} q_{2\mu} \right). \tag{4.10}$$

The Euler-Lagrange equations for $\lambda_{1\mu}$, $\lambda_{2\mu}$, and ξ_{μ} are respectively found to be

$$q_1^{\mu} = \dot{x}^{\mu}, \tag{4.11}$$
$$q_2^{\mu} = \dot{x}^{\mu}, \tag{4.12}$$
$$q_1^{\mu} = q_2^{\mu}, \tag{4.13}$$

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which are compatible with one another.

Substituting Eqs. (4.11) and (4.12) into Eq. (4.9) and noting

$$L_{\mathrm{A}}\left(q_1^\mu, \dot{q}_1^\mu, \lambda_{1\mu}; q_2^\mu, \dot{q}_2^\mu, \lambda_{2\mu}\right) = L_{\mathrm{A}}\left(\dot{x}^\mu, \ddot{x}^\mu, \lambda_{1\mu}; \dot{x}^\mu, \ddot{x}^\mu, \lambda_{2\mu}\right) = 0, \qquad (4.14)$$

we have

$$k\frac{d}{d\tau}\frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} = \frac{3}{2e}F^{\mu\nu}(x)\dot{x}_\nu + \frac{\dddot{x}_\perp^\mu}{\dot{x}^2} - \frac{3(\dot{x}\ddot{x})\ddot{x}_\perp^\mu}{(\dot{x}^2)^2} + \lambda_1^\mu - \xi^\mu. \qquad (4.15)$$

Similarly, substituting Eqs. (4.11) and (4.12) into Eq. (4.10) and using (4.14), we have

$$k\frac{d}{d\tau}\frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} = \frac{3}{2e}F^{\mu\nu}(x)\dot{x}_\nu + \frac{\dddot{x}_\perp^\mu}{\dot{x}^2} - \frac{3(\dot{x}\ddot{x})\ddot{x}_\perp^\mu}{(\dot{x}^2)^2} + \lambda_2^\mu - \xi^\mu. \qquad (4.16)$$

Comparing Eq. (4.15) and Eq. (4.16) leads to

$$\lambda_1^\mu = \lambda_2^\mu. \qquad (4.17)$$

Then we see that Eq. (4.4) is identically satisfied owing to $\partial_\mu F_{\nu\rho}(x)q_1^\nu q_2^\rho = \partial_\mu F_{\nu\rho}(x)\dot{x}^\nu \dot{x}^\rho = 0$. With $\Lambda^\mu := \lambda^\mu - \xi^\mu$ ($\lambda^\mu := \lambda_1^\mu = \lambda_2^\mu$), Eqs. (4.15) and (4.16) can be written together as a single equation

$$m\frac{d}{d\tau}\frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} = eF^{\mu\nu}(x)\dot{x}_\nu + \frac{2}{3}e^2\left(\frac{\dddot{x}_\perp^\mu}{\dot{x}^2} - \frac{3(\dot{x}\ddot{x})\ddot{x}_\perp^\mu}{(\dot{x}^2)^2}\right) + \frac{2}{3}e^2\Lambda^\mu, \qquad (4.18)$$

after the substitution of $k = 3m/2e^2$. This equation is completely the same as Eq. (3.23). In this way, it is established that the Lagrangian L_{A} also yields the Lorentz-Dirac equation with a source-like term.

5. Summary and discussion

We have presented two relativistic Lagrangians L_{D} and L_{A} and have demonstrated that the Euler-Lagrange equations derived from L_{D}, or those derived from L_{A}, together lead to the Lorentz-Dirac equation with a source-like term. This equation is a differential equation for x^μ having the inhomogeneous term $2e^2\Lambda^\mu/3$. Hence it follows that its solutions naturally depend on Λ^μ. The Lorentz-Dirac equation itself can be obtained in a particular situation such that $\Lambda^\mu = 0$. For this reason, L_{D} and L_{A} can simply be said to be the Lagrangians for the Lorentz-Dirac equation.

Contracting both sides of Eq. (4.18) with $\dot{x}^\mu$, we have the orthogonality condition

$$\dot{x}_\mu \Lambda^\mu = 0 \,. \tag{5.1}$$

This condition can be written as $\Lambda^0 = \boldsymbol{v} \cdot \boldsymbol{\Lambda}$, with $\boldsymbol{\Lambda} := (\Lambda^r)$ $(r = 1, 2, 3)$ and the velocity vector $\boldsymbol{v} := (dx^r/dx^0)$. Accordingly, the 4-vector (Λ^μ) can be expressed as $(\boldsymbol{v} \cdot \boldsymbol{\Lambda}, \boldsymbol{\Lambda})$. As can be seen from Eq. (4.18), the source-like term $2e^2 \Lambda^\mu/3$ is regarded as a component of the force 4-vector $(f^\mu) = (\boldsymbol{v} \cdot \boldsymbol{f}, \boldsymbol{f})$, provided that $\boldsymbol{f} := (2e^2/3)\boldsymbol{\Lambda}$ is identified with an external (non-Lorentzian) force acting on the charged particle. In this way, the source-like term can be treated as a component of the force 4-vector of an external force.

In the indirect formulation of the damped harmonic oscillator [28], a pair of two coordinate variables is introduced to describe the motion forward in time and that backward in time. A pair of q_1^μ and q_2^μ does not correspond to such a pair of coordinate variables . In fact, q_1^μ and q_2^μ are included even in the Lagrangian with an exponential damping function L_D. Also, $q_1^\mu = q_2^\mu = \dot{x}^\mu$ is eventually found from the Lagrangians L_D and L_A. For these reasons, q_1^μ and q_2^μ should simply be regarded as auxiliary variables useful for deriving the Lorentz-Dirac equation.

As has been emphasized above, L_D explicitly depends on the parameter τ, whereas L_A does not depend explicitly on τ. In a consistent quantization of the damped harmonic oscillator [25, 29, 30], the indirect formulation based on an explicitly time-independent Lagrangian is adopted rather than the direct formulation based on an explicitly time-dependent Lagrangian. Referring to this fact, we should choose L_A as a desirable Lagrangian when we consider quantum theory of a charged particle described by the Lorentz-Dirac equation. The Lagrangian and Hamiltonian formulations based on L_A and the subsequent quantization procedure are interesting issues that should be addressed in the future.

Appendix A. Lorentz-Dirac equation written in terms of an arbitrary world-line parameter

Let us recall the Lorentz-Dirac equation Eq. (1.2), i.e.,

$$m\frac{du^\mu}{dl} = eF^{\mu\nu}(x)u_\nu + \frac{2}{3}e^2\big(\delta^\mu{}_\nu - u^\mu u_\nu\big)\frac{d^2u^\nu}{dl^2}, \qquad (A.1)$$

which is written in terms of the proper time l. Noting that the infinitesimal proper time can be expressed as $dl = \sqrt{\dot{x}^2}d\tau$, we can show that

$$u^\mu := \frac{dx^\mu}{dl} = \frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}}, \qquad (A.2)$$

$$\frac{du^\mu}{dl} = \frac{\ddot{x}^\mu_\perp}{\dot{x}^2}, \qquad (A.3)$$

$$\frac{d^2u^\mu}{dl^2} = \frac{1}{\sqrt{\dot{x}^2}}\left[\frac{\dddot{x}^\mu_\perp}{\dot{x}^2} - \frac{3(\dot{x}\ddot{x})\ddot{x}^\mu_\perp}{(\dot{x}^2)^2} - \left(\ddot{x}^2 - \frac{(\dot{x}\ddot{x})^2}{\dot{x}^2}\right)\frac{\dot{x}^\mu}{(\dot{x}^2)^2}\right], \qquad (A.4)$$

with

$$\ddot{x}^\mu_\perp := \ddot{x}^\mu - \frac{\dot{x}\ddot{x}}{\dot{x}^2}\dot{x}^\mu, \qquad \dddot{x}^\mu_\perp := \dddot{x}^\mu - \frac{\dot{x}\dddot{x}}{\dot{x}^2}\dot{x}^\mu. \qquad (A.5)$$

Here,

$$\dot{x}^\mu := \frac{dx^\mu}{d\tau}, \qquad \ddot{x}^\mu := \frac{d^2x^\mu}{d\tau^2}, \qquad \dddot{x}^\mu := \frac{d^3x^\mu}{d\tau^3}, \qquad (A.6)$$

$$\dot{x}^2 := \dot{x}_\mu\dot{x}^\mu, \qquad \ddot{x}^2 := \ddot{x}_\mu\ddot{x}^\mu, \qquad \dot{x}\ddot{x} := \dot{x}_\mu\ddot{x}^\mu, \qquad \dot{x}\dddot{x} := \dot{x}_\mu\dddot{x}^\mu. \qquad (A.7)$$

Substituting Eqs. (A.2) and (A.4) into Eq. (A.1), we obtain

$$m\frac{d}{d\tau}\frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} = eF^{\mu\nu}(x)\dot{x}_\nu + \frac{2}{3}e^2\left(\frac{\dddot{x}^\mu_\perp}{\dot{x}^2} - \frac{3(\dot{x}\ddot{x})\ddot{x}^\mu_\perp}{(\dot{x}^2)^2}\right). \qquad (A.8)$$

This is the Lorentz-Dirac equation written in terms of an arbitrary world-line parameter τ. We can directly derive Eq. (A.8) by evaluating the reaction force due to the particle's own electromagnetic radiation without adopting the proper-time gauge $\tau = l$.

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